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Symmetries of quantum systems: a partial inner product space approach

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Abstract
We first give a quick survey of the realization of symmetries of quantum systems in the various formalisms of quantum mechanics: traditional (Hilbert space), algebraic (C*-algebras), rigged Hilbert spaces, *-algebras of unbounded operators, partial *-algebras of closable operators. Then we describe in some detail the concept of partial inner product spaces (PIP-spaces) and operators on them. Finally, we examine various classes of operators on PIP-spaces that allow a correct realization of symmetries.

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1. How to describe a quantum system and its symmetries?

The mathematical description of a quantum system has evolved considerably since the creation of quantum mechanics in the 1920s. This is indeed a striking example of cross-fertilization between two disciplines. As a matter of fact, the whole edifice rests on two basic principles:

(i) The *superposition principle*, which implies that the set of states of the system has a linear structure;
(ii) The notion of *transition amplitude*, given by an inner product: $A(\psi_1 \to \psi_2) = \langle \psi_2 | \psi_1 \rangle$. The latter in turn yields transition probabilities by $P(\psi_1 \to \psi_2) = |\langle \psi_2 | \psi_1 \rangle|^2$.

1.1. The traditional approach

Combining these two basic principles implies that the set of states of the system is a positive definite inner product space, that is, a pre-Hilbert space. Then mathematical efficiency leads to the traditional Hilbert space formulation of Schrödinger, Dirac, von Neumann, etc, namely,

- *States* are represented by rays in a Hilbert space $\mathfrak{H}$;
- *Observables* are represented by self-adjoint operators in $\mathfrak{H}$.

In this context, a *symmetry* is defined as a bijection between states that preserves the absolute values of all transition amplitudes. According to Wigner, a symmetry τ is realized by a unitary or an anti-unitary operator in $\mathfrak{H}$ [24]. Then, if the system admits a *symmetry group* $\{\tau_g, g \in G\}$, with G a Lie group, the latter is realized by a strongly continuous unitary (projective) representation $U(g)$ of G in $\mathfrak{H}$ (Wigner–Bargmann) [6, 24].

In order to free the theory from particular realizations, linked to specific models, and to focus on the basic structure, namely, the algebra of observables and states on it, Haag and Kastler [16] introduced in 1960 an abstract version of the theory. Here observables are realized by self-adjoint elements of a C*-algebra $\mathfrak{A}$ and states by (normalized) continuous linear functionals on $\mathfrak{A}$. Then a symmetry τ is realized by a *-automorphism σ of $\mathfrak{A}$ and a symmetry group $\{\tau_g, g \in G\}$ by a continuous *-automorphism group $\{\sigma_g, g \in G\}$ of $\mathfrak{A}$. This more abstract language, exploiting the deep mathematical theories of C*-algebras and von Neumann algebras, soon became the standard approach to the mathematically rigorous description of physical systems with infinitely many degrees of freedom.

In this framework, a concrete Hilbert space representation is obtained via the Gel'fand–Naĭmark–Segal (GNS) construction. Thus we are back to the traditional approach, with observables represented by *bounded* operators. As for symmetries, the question is now whether they are unitarily implemented in the GNS Hilbert space.

1.2. The rigged Hilbert space approach

Although standard, the traditional approach still has difficulties. Unbounded operators are often more natural than bounded ones (e.g. representatives of a Lie algebra, such as symmetry generators), but then one may have domain problems. Also not all self-adjoint operators can be interpreted as physical observables. Neither do all states play the same role. Indeed, there are '*physical*' states, that can actually be prepared, and '*generalized*' states, associated with quantum measurements.

Assume all 'relevant' observables have a common, dense, invariant domain in $\mathfrak{H}$. Then one gets a *rigged Hilbert space (RHS)*

$$\Phi \subset \mathfrak{H} \subset \Phi^\times, \tag{1.1}$$

where Φ is the set of all physical states and $\Phi^\times$, the set of continuous antilinear functionals on Φ, consists of the generalized states associated with measurement devices [1, 9, 10].

The problem, of course, is how to build Φ. A solution, introduced by Roberts [21], is to start from a distinguished set O of *labeled* observables with a physical interpretation (how does one measure it?) *and* a mathematical definition (as a self-adjoint operator in $\mathfrak{H}$). The elements of O, which characterize the system (physics): position, momentum, energy (Hamiltonian), ..., are supposed to have a common, dense, invariant domain $\mathcal{D}$ in $\mathfrak{H}$ (mathematics). If one equips $\mathcal{D}$ with a suitable (intrinsic) topology, one obtains a RHS (1.1) defined by the system.

The simplest example in nonrelativistic quantum mechanics is that of a particle, either free or in a nice potential V. The labeled observables are position $\mathbf{q}$, momentum $\mathbf{p}$ and energy $H = -\mathbf{p}^2/2m + V$. The corresponding RHS is $\mathcal{S} \subset L^2(\mathbb{R}^3) \subset \mathcal{S}^\times$, where $\mathcal{S}$ is Schwartz's space of smooth fast decaying functions and $\mathcal{S}^\times$ is the space of tempered distributions.

What about symmetries in this RHS approach [1]? In the standard realization, a symmetry group G is realized by a unitary representation U of G in $\mathfrak{H}$. For consistency, one has to require that U maps physical states into physical states and similarly for the measuring devices. Thus one should have two other realizations of U, in addition to U itself, which acts in $\mathfrak{H}$, namely:

- U_Φ acting in Φ (*active* point of view);
- $U_\Phi^\times$ acting in $\Phi^\times$ (*passive* point of view).

The equivalence of the two points of view is manifested by the requirement that U_Φ and $U_\Phi^\times$ are contragradient of each other, that is,

$$\left\langle U_\Phi^\times(g)F\middle|\phi\right\rangle = \langle F|U_\Phi(g^{-1})\phi\rangle, \qquad \forall\, g \in G, \phi \in \Phi, F \in \Phi^\times,$$

or, equivalently,

$$\left\langle U_\Phi^\times(g)F\middle|U_\Phi(g)\phi\right\rangle = \langle F|\phi\rangle, \qquad \forall\, g \in G, \phi \in \Phi, F \in \Phi^\times.$$

This corresponds to the unitarity of U acting in $\mathfrak{H}$:

$$\langle U(g)f|U(g)h\rangle = \langle f|h\rangle, \qquad \forall\, g \in G,\ f, h \in \mathfrak{H}.$$

This definition implies that $U_\Phi^\times$ is an extension of both U_Φ and U, as it should in view of (1.1).

As in the standard approach, there is a corresponding abstract version of the theory, in which the observable algebra $\mathfrak{A}$ is assumed to be a **-algebra of unbounded operators*. The mathematical technology is available thanks to the work of many authors (Powers [20], Lassner [18], Schmüdgen [23], etc), including the GNS construction and the description of *-automorphism groups and derivations of $\mathfrak{A}$.

1.3. The partial operator algebra approach

However, more difficulties may arise. Indeed, it is not always possible, or convenient, to find an *invariant* common dense domain for all observables of the system. However, if one drops this requirement, the product of two such operators A, B need no longer be defined. Namely, AB makes sense only if the range of B is contained in the domain of A. This suggests extending one step further the description of $\mathfrak{A}$, and taking it as a *partial *-algebra of closable operators* on $\mathfrak{H}$. Once again, the mathematical technology is available, including the GNS construction and the notions of *-automorphism groups and derivations [5]. In fact most concepts familiar in the theory of C*-algebras extend to this wider framework, but at the price of severe technical complications. Thus this approach, while intellectually satisfying, is not directly applicable for quantum mechanics, we need something simpler and more natural. Our answer to that query is the notion of *partial inner product spaces* (PIP-spaces), which we will describe in the following.

2. Partial inner product spaces

Let us go back to the basic principles stated in section 1. From the discussion made above, it is clear that not all states are equally accessible. Hence the transition is possible only between *certain pairs* of states. Thus one should use a *partial* inner product to modelize transition amplitudes, and one is led to a PIP-space, a structure introduced some time ago by Grossmann and the present author [2, 4].

2.1. Basic definitions

The basic question may be stated as follows: given a vector space V and two vectors $f, g \in V$, when does their inner product make sense? A way of formalizing the answer is given by the idea of *compatibility*.

Definition 2.1. *A linear compatibility relation on a vector space V is a symmetric binary relation $\#$ which preserves linearity:*

$$f\#g \iff g\#f, \qquad \forall\, f, g \in V,$$
$$f\#g,\ f\#h \implies f\#(\alpha g + \beta h), \qquad \forall\, f, g, h \in V, \quad \forall\, \alpha, \beta \in \mathbb{C}.$$

As a consequence, for every subset $S \subset V$, the set $S^{\#} = \{g \in V : g\#f, \forall f \in S\}$ is a vector subspace of V and one has

$$S^{\#\#} = (S^{\#})^{\#} \supseteq S, \qquad S^{\#\#\#} = S^{\#}.$$

Thus one gets the following equivalences:

$$\begin{aligned} f\#g \iff& f \in \{g\}^{\#} \iff \{f\}^{\#\#} \subseteq \{g\}^{\#} \\ \iff& g \in \{f\}^{\#} \iff \{g\}^{\#\#} \subseteq \{f\}^{\#}. \end{aligned} \tag{2.1}$$

From now on, we will call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$ and denote by $\mathcal{F}(V, \#)$ the family of all assaying subsets of V, ordered by inclusion. Let F be the isomorphy class of $\mathcal{F}$, that is, $\mathcal{F}$ considered as an abstract partially ordered set. Elements of F will be denoted by $r, q, \ldots$, and the corresponding assaying subsets $V_r, V_q, \ldots$. By definition, $q \leqslant r$ if and only if $V_q \subseteqq V_r$. We also write $V_{\overline{r}} = V_r^{\#}, r \in F$. Thus the relations (2.1) mean that $f\#g$ if and only if there is an index $r \in F$ such that $f \in V_r, g \in V_{\overline{r}}$. In other words, vectors should not be considered individually, but only in terms of assaying subspaces, which are the building blocks of the whole structure.

It is easy to see that the map $S \mapsto S^{\#\#}$ is a closure, in the sense of universal algebra, so that the assaying subspaces are precisely the 'closed' subsets. Therefore one has the following standard result [8].

Theorem 2.2. *The family $\mathcal{F}(V, \#) = \{V_r, r \in F\}$ of all assaying subspaces, ordered by inclusion, is a complete involutive lattice, under the following operations:*

$$\bigwedge_{j\in J} V_j = \bigcap_{j\in J} V_j, \qquad \bigvee_{j\in J} V_j = \left(\sum_{j\in J} V_j\right)^{\#\#}, \qquad \textit{for any subset } J \subset F, \tag{2.2}$$

and the involution $V_r \leftrightarrow V_{\overline{r}} = (V_r)^{\#}$. Moreover, the involution is a lattice anti-isomorphism, that is,

$$(V_r \wedge V_s)^{\#} = V_{\overline{r}} \vee V_{\overline{s}}, \qquad (V_r \vee V_s)^{\#} = V_{\overline{r}} \wedge V_{\overline{s}}.$$

The smallest element of $\mathcal{F}(V, \#)$ is $V^{\#} = \bigcap_r V_r$ and the greatest element is $V = \bigcup_r V_r$. By definition, the index set F is also a complete involutive lattice; for instance,

$$(V_{p\wedge q})^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}.$$

Definition 2.3. *A partial inner product on $(V, \#)$ is a Hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors. A partial inner product space (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product.*

Note that the partial inner product is not required to be positive definite. Nevertheless, the partial inner product clearly defines a notion of *orthogonality* : $f \perp g$ if and only if $f\#g$ and $\langle f|g\rangle = 0$.

Definition 2.4. *The PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is nondegenerate if $(V^{\#})^{\perp} = \{0\}$, that is, if $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$.*

From now on, we will assume that our PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is nondegenerate. As a consequence, $(V^{\#}, V)$ and every couple $(V_r, V_{\overline{r}}), r \in F$, are a dual pair in the sense of topological vector spaces [17, 22]. We also assume that the partial inner product is positive definite.

Now, one wants the topological structure to match the algebraic structure. This means, in particular, that the topology $\mathrm{t}(V_r)$ of V_r must be such that the dual of V_r is precisely $V_{\overline{r}}$, that is, $\mathrm{t}(V_r)$ is a topology of the dual pair $\langle V_r, V_{\overline{r}}\rangle$. Therefore $\mathrm{t}(V_r)$ must be finer than the weak topology $\sigma(V_r, V_{\overline{r}})$ and coarser than the Mackey topology $\tau(V_r, V_{\overline{r}})$:

$$\sigma(V_r, V_{\overline{r}}) \preceq \mathrm{t}(V_r) \preceq \tau(V_r, V_{\overline{r}}).$$

Assumption. *From here on, we will assume that every V_r carries its Mackey topology $\tau(V_r, V_{\overline{r}})$.*

This choice has two interesting consequences. First, if $V_r[\mathrm{t}(V_r)]$ is a Hilbert space or a reflexive Banach space, then $\tau(V_r, V_{\overline{r}})$ coincides with the norm topology. Next, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has a dense range. In particular, $V^\#$ is dense in every V_r.

2.2. Examples

Let us give two simple examples of PIP-spaces.

(i) *Sequence spaces*
Let V be the space ω of *all* complex sequences $x = (x_n)$ and define on it (i) a compatibility relation by $x\#y \Leftrightarrow \sum_{n=1}^{\infty} |x_n y_n| < \infty$; (ii) a partial inner product $\langle x|y\rangle = \sum_{n=1}^{\infty} \overline{x_n} y_n \ (x\#y)$.
Then $\omega^\# = \varphi$, the space of finite sequences, and the complete lattice $\mathcal{F}(\omega, \#)$ consists of Köthe's perfect sequence spaces [[17], section 30]. Among these, typical assaying subspaces are the weighted Hilbert spaces

$$\ell^2(r) = \left\{ (x_n) : \sum_{n=1}^{\infty} |x_n|^2 r_n^{-1} < \infty \right\}, \tag{2.3}$$

where $r = (r_n), r_n > 0$, is a sequence of positive numbers. The involution is $\ell^2(r) \leftrightarrow \ell^2(\overline{r}) = \ell^2(r)^\times$, where $\overline{r}_n = 1/r_n$. In addition, there is a central, self-dual Hilbert space, namely, $\ell^2(1) = \ell^2(\overline{1}) = \ell^2$, where 1 denotes the unit sequence, $r_n = 1, \forall n$.

(ii) *Spaces of locally integrable functions*
Now let V be $L^1_{\mathrm{loc}}(\mathbb{R}, dx)$, the space of Lebesgue measurable functions, integrable over compact subsets. Define a compatibility relation on it by $f\#g \Leftrightarrow \int_{\mathbb{R}} |f(x)g(x)|\,\mathrm{d}x < \infty$ and a partial inner product $\langle f|g\rangle = \int_{\mathbb{R}} \overline{f(x)} g(x)\,\mathrm{d}x \ (f\#g)$.
Then $V^\# = L^\infty_{\mathrm{c}}(\mathbb{R})$, the space of bounded measurable functions of compact support. The complete lattice $\mathcal{F}\big(L^1_{\mathrm{loc}}, \#\big)$ consists of Köthe function spaces [12, 13]. Here again, typical assaying subspaces are weighted Hilbert spaces

$$L^2(r) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x) : \int_{\mathbb{R}} |f(x)|^2 r(x)^{-1}\,\mathrm{d}x < \infty \right\}, \tag{2.4}$$

with $r, r^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x), r(x) > 0$ a.e. The involution is $L^2(r) \leftrightarrow L^2(\overline{r})$, with $\overline{r} = r^{-1}$, and the central, self-dual Hilbert space is $L^2(\mathbb{R}, \mathrm{d}x)$.

2.3. Lattices of Hilbert or Banach spaces

From the previous examples, we learn that $\mathcal{F}(V, \#)$ is a huge lattice (it is complete!) and that assaying subspaces may be complicated, such as Fréchet spaces, nonmetrizable spaces, etc. This situation suggests choosing a sublattice $\mathcal{I} \subset \mathcal{F}$, indexed by I, such that

(i) $\mathcal{I}$ is *generating*:

$$f \# g \Leftrightarrow \exists\, r \in I \text{ such that } f \in V_r,\, g \in V_{\overline{r}}; \tag{2.5}$$

(ii) every $V_r, r \in I$, is a Hilbert space or a reflexive Banach space;

(iii) there is a unique self-dual, Hilbert, assaying subspace $V_o = V_{\overline{o}}$.

In that case, the structure $V_I := (V, \mathcal{I}, \langle\cdot|\cdot\rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS). Both types are particular cases of the so-called indexed PIP-spaces [4], but they are sufficient for our present purposes. Note that $V^\#, V$ themselves usually do *not* belong to the family $\{V_r, r \in I\}$, but they can be recovered as $V^\# = \bigcap_{r\in I} V_r$, $V = \sum_{r\in I} V_r$.

In the LBS case, the lattice structure takes the following form:

- $V_{p\wedge q} = V_p \cap V_q$, with the *projective* norm $\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q$;
- $V_{p\vee q} = V_p + V_q$, with the *inductive* norm $\|f\|_{p\vee q} = \inf_{f=g+h}(\|g\|_p + \|h\|_q)$, $g \in V_p$, $f \in V_q$.

These norms are usual in interpolation theory [7]. In the LHS case, one takes similar definitions with squared norms, in order to get Hilbert norms throughout.

2.4. Examples

We list a series of concrete examples of LBSs. For simplicity, we restrict ourselves to one dimension, although most spaces may be defined on $\mathbb{R}^n$, $n > 1$, as well.

(i) Scales of Hilbert or Banach spaces

(a) *The Lebesgue L^p spaces on a finite interval*, e.g. $\mathcal{I} = \{L^p([0, 1], dx), 1 \leqslant p \leqslant \infty\}$

$$L^\infty \subset \ldots \subset L^{\overline{q}} \subset L^{\overline{r}} \subset \ldots \subset L^2 \subset \ldots \subset L^r \subset L^q \subset \ldots \subset L^1, \tag{2.6}$$

where $1 < q < r < 2$. Here L^q and $L^{\overline{q}}$ are dual to each other $(1/q + 1/\overline{q} = 1)$, and similarly $L^r, L^{\overline{r}} (1/r + 1/\overline{r} = 1)$. By the Hölder inequality, the (L^2) inner product

$$\langle f|g\rangle = \int_0^1 \overline{f(x)} g(x)\, \mathrm{d}x \tag{2.7}$$

is well-defined if $f \in L^q, g \in L^{\overline{q}}$. However, it is *not* well-defined for two arbitrary functions $f, g \in L^1$. Take, for instance, $f(x) = g(x) = x^{-1/2}$. Then $f \in L^1$, but $fg = f^2 \notin L^1$. Thus, on L^1, (2.7) defines only a *partial* inner product. The same result holds for any compact subset of $\mathbb{R}$ instead of [0,1].

The corresponding lattice completion is obtained by adding 'nonstandard' spaces, such as

$$L^{p-} = \bigcap_{1<q<p} L^q \text{ (non-normable Fréchet)}, \qquad L^{p+} = \bigcup_{p<q<\infty} L^q \text{ (nonmetrizable)}.$$

(b) *The scale of Hilbert spaces built on the powers of a positive self-adjoint operator $A \geqslant 1$ in a Hilbert space $\mathfrak{H}_0$*:

Let $\mathfrak{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathfrak{H}_{-n} = \mathfrak{H}_n^\times$ (conjugate dual):

$$\mathfrak{H}_\infty(A) := \bigcap_n \mathfrak{H}_n \subset \ldots \subset \mathfrak{H}_2 \subset \mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-1} \subset \mathfrak{H}_{-2} \ldots \subset \mathfrak{H}_{-\infty}(A) := \bigcup_n \mathfrak{H}_n. \tag{2.8}$$

Note that here the index n may be integer or real, the link between the two cases being established by the spectral theorem for self-adjoint operators. Here again the inner product of $\mathfrak{H}_0$ extends to each pair $\mathfrak{H}_n, \mathfrak{H}_{-n}$, but on $\mathfrak{H}_{-\infty}(A)$ it yields only a *partial* inner product. The following examples, all three in $\mathfrak{H}_0 = L^2(\mathbb{R}, \mathrm{d}x)$, are standard:

- $(A_{\mathrm{p}}f)(x) = (1 + x^2)^{1/2} f(x)$;
- $(A_{\mathrm{m}}f)(x) = \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} f(x)$;
- $(A_{\mathrm{osc}}f)(x) = \left(1 + x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) f(x)$.

(The notation is suggested by the operators of position, momentum and harmonic oscillator energy in quantum mechanics, respectively.) In the case of A_{m}, the intermediate spaces are the Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$ or $\mathbb{R}$. Note that both $\mathfrak{H}_\infty(A_{\mathrm{p}}) \cap \mathfrak{H}_\infty(A_{\mathrm{m}})$ and $\mathfrak{H}_\infty(A_{\mathrm{osc}})$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay, and $\mathfrak{H}_{-\infty}(A_{\mathrm{osc}})$ with the space $\mathcal{S}^\times(\mathbb{R})$ of tempered distributions.

The lattice completion of this scale is similar to that of the previous one.

(ii) Sequence spaces

In ω, we may take the lattice $\mathcal{I} = \{\ell^2(r)\}$ of the weighted Hilbert spaces defined in (2.3), with

- infimum: $\ell^2(p \wedge q) = \ell^2(p) \wedge \ell^2(q) = \ell^2(r)$, $r_n = \min(p_n, q_n)$,
- supremum: $\ell^2(p \vee q) = \ell^2(p) \vee \ell^2(q) = \ell^2(s)$, $s_n = \max(p_n, q_n)$,
- duality: $\ell^2(p \wedge q) \leftrightarrow \ell^2(\overline{p} \vee \overline{q})\ell^2(p \vee q) \leftrightarrow \ell^2(\overline{p} \wedge \overline{q})$

(the norms above are equivalent to the projective and inductive norms, respectively).

(iii) Spaces of locally integrable functions

In $L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x)$, we may take the lattice $\mathcal{I} = \{L^2(r)\}$ of the weighted Hilbert spaces defined in (2.3), with

- infimum: $L^2(p \wedge q) = L^2(p) \wedge L^2(q) = L^2(r)$, $r(x) = \min(p(x), q(x))$,
- supremum: $L^2(p \vee q) = L^2(p) \vee L^2(q) = L^2(s)$, $s(x) = \max(p(x), q(x))$,
- duality: $L^2(p \wedge q) \leftrightarrow L^2(\overline{p} \vee \overline{q})$, $L^2(p \vee q) \leftrightarrow L^2(\overline{p} \wedge \overline{q})$.

(iv) The spaces $L^p(\mathbb{R}, \mathrm{d}x)$, $1 < p < \infty$

The spaces $L^p(\mathbb{R}, \mathrm{d}x)$, $1 < p < \infty$ do not constitute a scale, since one has only the inclusions $L^p \cap L^q \subset L^s$, $p < s < q$. Thus one has to consider the lattice they generate, with the following lattice operations:

- $L^p \wedge L^q = L^p \cap L^q$, with the projective norm ;
- $L^p \vee L^q = L^p + L^q$, with the inductive norm ;
- For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive Banach spaces and $(L^p \wedge L^q)' = L^{\overline{p}} \vee L^{\overline{q}}$, $(L^p \vee L^q)' = L^{\overline{p}} \wedge L^{\overline{q}}$.

The result is that one gets a genuine lattice of Banach spaces, reflexive for $1 < p, q < \infty$.

3. Operators on PIP-spaces

3.1. Basic idea

As already mentioned, the basic idea of (indexed) PIP-spaces is that vectors should not be considered individually, but only in terms of the subspaces V_r ($r \in F$ or $r \in I$), the building blocks of the structure, see (2.5). Correspondingly, an operator on a PIP-space should be defined in terms of assaying subspaces only, with the proviso that only bounded operators between Hilbert or Banach spaces are allowed. Thus an operator is a *coherent collection* of bounded operators. More precisely:

Definition 3.1. *Given a LHS or LBS $V_I = \{V_r, r \in I\}$, an operator on V_I is a map from a subset $\mathcal{D} \subseteq V$ into V, where*

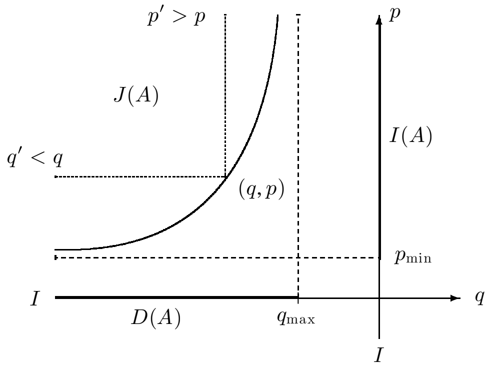

Figure 1. Characterization of an operator in the case of a scale.

(i) $\mathcal{D}$ is a nonempty union of assaying subsets of V_I;

(ii) for every assaying subset V_q contained in $\mathcal{D}$, there exists a $p \in I$ such that the restriction of A to V_q is linear and continuous into V_p (we denote this restriction by A_{pq});

(iii) A has no proper extension satisfying (i) and (ii), i.e., it is maximal.

(A proper extension of A satisfying (i) and (ii) would be a map A' defined on a union of assaying subsets $\mathcal{D}' \supset \mathcal{D}$, coinciding with A on $\mathcal{D}$, linear and continuous on every assaying subset in its domain.)

The linear bounded operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. In terms of the latter, the operator A may be characterized by the set $J(A) := \{(q, p) \in I \times I : A_{pq} \text{exists}\}$. Thus the operator A may be identified with the collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q, p) \in J(A)\}.$$

We also need the two sets obtained by projecting $J(A)$ on the 'coordinate' axes, namely,

$$D(A) := \mathrm{pr}_1 J(A) = \{q \in I : \text{there is a } p \text{ such that} A_{pq} \text{ exists}\},$$

$$I(A) := \mathrm{pr}_2 J(A) = \{p \in I : \text{there is a } q \text{ such that} A_{pq} \text{ exists}\},$$

where pr_1 and pr_2 denote the first and the second projection, respectively, of a set in the Cartesian product $I \times I$.

The following properties are immediate:

- $D(A)$ is an initial subset of I: if $q \in D(A)$ and $q' < q$, then $q' \in D(A)$, and $A_{pq'} = A_{pq} E_{qq'}$, where $E_{qq'}$ is a representative of the unit operator (this is what we mean by a 'coherent' collection).
- $I(A)$ is a final subset of I: if $p \in I(A)$ and $p' > p$, then $p' \in I(A)$ and $A_{p'q} = E_{p'p} A_{pq}$.
- $J(A) \subset D(A) \times I(A)$, with strict inclusion in general.

We denote by $\mathrm{Op}(V_I)$ the set of all operators on V_I. Of course, a similar definition may be given for operators $A : V_I \to Y_K$ between two LHSs or LBSs.

In the case of a scale, all the sets described above, which characterize an operator A, are displayed on the diagram of figure 1.

3.2. Algebraic operations on operators

Since $V^\#$ is dense in V_r, for every $r \in I$, an operator may be identified with a separately continuous sesquilinear form on $V^\# \times V^\#$. Indeed, the restriction of any representative A_{pq} to $V^\# \times V^\#$ is such a form, and all these restrictions coincide. Equivalently, an operator may be identified with a continuous linear map from $V^\#$ into V (continuity with respect to the respective Mackey topologies).

But the idea behind the notion of operator is to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint A^*:* every $A \in \mathrm{Op}(V_I)$ has a unique adjoint $A^* \in \mathrm{Op}(V_I)$, defined by the relation

$$\langle A^* x|y\rangle = \langle x|Ay\rangle, \qquad \text{for} \quad y \in V_r, r \in D(A), \quad \text{and} \quad x \in V_{\overline{s}}, s \in I(A),$$

that is, $(A^*)_{\overline{r}\overline{s}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint). In other words, $J(A^*) = \{(\overline{s}, \overline{r}) \in I \times I : (r, s) \in J(A)\}$.

It follows that $A^{**} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii) of definition 3.1.

(ii) *Partial multiplication:* AB is defined if and only if there is a $q \in I(B) \cap D(A)$, that is, if and only if there is a continuous factorization through some V_q :

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \qquad \text{i.e.,} \qquad (AB)_{sr} = A_{sq} B_{qr}.$$

It is worth noting that, for a LHS/LBS, the domain $\mathcal{D}$ of an operator is always a vector subspace of V (this is not true for a general PIP-space). Therefore, $\mathrm{Op}(V_I)$ is a vector space and a *partial *-algebra* [5].

The concept of PIP-space operator is very simple, yet it is a far reaching generalization of bounded operators. It allows indeed to treat on the same footing all kinds of operators, from bounded ones to very singular ones. By this, we mean the following, loosely speaking. Take

$$V_r \subset V_o \simeq V_{\overline{o}} \subset V_s \quad (V_o = \text{Hilbert space}).$$

Three cases may arise:

- if A_{oo} exists, then A corresponds to a *bounded* operator $V_o \to V_o$;
- if A_{oo} does not exist, but only $A_{or} : V_r \to V_o$, with $r < o$, then A corresponds to an *unbounded* operator, with Hilbert space domain containing V_r;
- if no A_{or} exists, but only $A_{sr} : V_r \to V_s$, with $r < o < s$, then A corresponds to a *singular* operator, with Hilbert space domain possibly reduced to $\{0\}$.

A nice application of this machinery is a rigorous analysis of singular quantum Hamiltonians (e.g. rigorous versions of the Kronig–Penney crystal model or of δ interactions) [15].

4. Special classes of operators on PIP-spaces

Exactly as for Hilbert or Banach spaces, one may define various types of operators between PIP-spaces, in particular LBS/LHS.

4.1. Regular operators

An operator A is called *regular* if $D(A) = I(A) = I$ or, equivalently, if $A : V^\# \to V^\#$ and $A : V \to V$ continuously for the respective Mackey topologies. This notion depends only on the pair $(V^\#, V)$, not on the particular compatibility $\#$. The set of all regular operators $V_I \to V_I$ is denoted by $\mathrm{Reg}(V_I)$. Thus a regular operator may be multiplied both on the left and on the right by an arbitrary operator. Clearly, the set $\mathrm{Reg}(V_I)$ is a *-algebra [19].

We give two examples.

(1) If $V = \omega$, $V^\# = \varphi$, then $\mathrm{Op}(\omega)$ consists of arbitrary infinite matrices and $\mathrm{Reg}(\omega)$ of infinite matrices with finite rows and finite columns.
(2) If $V = \mathcal{S}^\times$, $V^\# = \mathcal{S}$, then $\mathrm{Op}(\mathcal{S}^\times)$ consists of arbitrary tempered kernels, while $\mathrm{Reg}(\mathcal{S}^\times)$ contains those kernels that can be extended to $\mathcal{S}^\times$ and map $\mathcal{S}$ into itself. A nice example is the Fourier transform.

*4.2. *-Algebras of operators*

(i) An operator A is called *totally regular* if $J(A)$ contains the diagonal of $I \times I$, i.e., A_{rr} exists for every $r \in I$ or A maps every V_r into itself continuously. We denote by $\mathfrak{A}(V_I)$ the set of all totally regular operators $V_I \to V_I$. Clearly $\mathfrak{A}(V_I)$ is a *-algebra.
(ii) We define $\mathfrak{B}(V_I) = \{A \in \mathfrak{A}(V_I) : \|A\|_{\mathfrak{B}} := \sup_{r\in I} \|A_{rr}\|_r < \infty\}$. Then one can show that $\mathfrak{B}(V_I)$ is a Banach algebra for the norm $\|\cdot\|_{\mathfrak{B}}$ [19].
(iii) In the LHS case, we define $\mathfrak{C}(V_I) = \{A \in \mathfrak{B}(V_I) : \forall r \in I, A_{rr} = u_{r\bar{r}} A_{\bar{r}\bar{r}} u_{\bar{r}} r\}$, where $u_{r\bar{r}} : V_r \to V_{\bar{r}}$ is the Riesz unitary map between the Hilbert space V_r and its conjugate dual $V_{\bar{r}}$. Then one shows that $\mathfrak{C}(V_I)$ is a von Neumann algebra.

Thus we get

$$\mathfrak{A}(V_I) \supset \mathfrak{B}(V_I) \supset \mathfrak{C}(V_I).$$

We know that a von Neumann algebra is always generated by its projections. As explained in the next section, it turns out that the Hilbert space definition of a projection extends to PIP-spaces, with similar properties, including the one-to-one correspondence with appropriate subspaces. Thus the interest of $\mathfrak{C}(V_I)$ (and the motivation for introducing it) is to show, in the LHS case, the existence of sufficiently many projections and/or subspaces.

4.3. Orthogonal projections

An operator P on a nondegenerate PIP-space V, resp. a LBS/LHS V_I, is an *orthogonal projection* if P is totally regular and $P^2 = P = P^*$ [3]. We denote by $\mathrm{Proj}(V)$ the set of all orthogonal projections in V and similarly for $\mathrm{Proj}(V_I)$. It turns out that $\mathrm{Proj}(V)$, resp. $\mathrm{Proj}(V_I)$, is a partially ordered set, as in a Hilbert space. However, it is a lattice only under additional conditions, yet to be determined, the problem is still open.

These projection operators enjoy several properties similar to those of Hilbert space projectors. Two of them are of special interest in the present context.

(i) Given a nondegenerate PIP-space V, there is a natural notion of PIP-subspace, called *orthocomplemented*, which guarantees that such a subspace W of V is again a nondegenerate PIP-space with the induced compatibility relation and the restriction of the partial inner product. Then the basic theorem about projections states that a PIP-subspace W of V is orthocomplemented if and only if W is the range of an orthogonal projection $P \in \mathrm{Proj}(V)$, i.e., $W = PV$. Then $V = W \oplus Z$, where Z is another orthocomplemented PIP-subspace.

(ii) An orthogonal projection P is of finite rank if and only if $W = \mathrm{Ran}P \subset V^{\#}$ and $W \cap W^{\perp} = \{0\}$ (this second condition is, of course, superfluous when the partial inner product is positive definite, as it is the case here).

Property (ii) has important consequences for the structure of bases. First we recall that a sequence $\{e_n, n = 1, 2, \ldots\}$ of vectors in a Banach space E is a *Schauder basis* if, for every $f \in E$, there exists a unique sequence of coefficients $\{c_k, k = 1, 2, \ldots\}$ such that $\lim_{m\to\infty} \left\| f - \sum_{k=1}^{m} c_k e_k \right\| = 0$. Then one may write

$$f = \sum_{k=1}^{\infty} c_k e_k. \tag{4.1}$$

The basis is *unconditional* if the series (4.1) converges unconditionally in E (i.e., it keeps converging after an arbitrary permutation of its terms).

For instance, a standard problem is to find a sequence of functions that is an unconditional basis for *all* the spaces $L^p(\mathbb{R})$, $1 < p < \infty$. In the PIP-space language, this statement means that the basis vectors must belong to $V^{\#} = \cap_{1<p<\infty} L^p(\mathbb{R})$. Also, since (4.1) means that f may be approximated by finite sums, property (ii) of orthogonal projections implies that all the basis vectors must belong to $V^{\#}$. Actually, some wavelet bases solve the L^p basis problem [11].

Similar considerations apply to frames. We recall here that a *frame* in a Hilbert space $\mathfrak{H}$ is a family of vectors $\{e_j, j \in J\}$ for which there exist two constants $0 < \mathsf{m} \leqslant \mathsf{M} < \infty$ such that

$$\mathsf{m}\|f\|^2 \leqslant \sum_{j\in J} |\langle e_j|f\rangle|^2 \leqslant \mathsf{M}\|f\|^2. \tag{4.2}$$

The frame is called *tight* if $\mathsf{m} = \mathsf{M}$. It turns out that frames and, in particular, tight frames are often good substitutes for orthogonal bases for representing an arbitrary vector by a fast converging expansion $f = \sum_{j\in J} c_j e_j$ (e.g. in the theory of wavelets [11]).

4.4. Homomorphisms, isomorphisms, and all that

An operator $A : V_I \to V_I$ is called a *homomorphism* if $\mathrm{pr}_1(J(A) \cap \overline{J(A)}) = I$ and $\mathrm{pr}_2(J(A) \cap \overline{J(A)}) = I$, where $\overline{J(A)} = \{(\overline{r}, \overline{s}) : (r, s) \in J(A)\}$. In words, for every $r \in I$, there exists $r' \in I$ such that $(r, r') \in J(A)$ and $(\overline{r}, \overline{r}') \in J(A)$, and for every $r' \in I$, there exists $r \in I$ with the same property. The second condition means that, if A is a homomorphism, A^* is also one. We denote by $\mathrm{Hom}(V_I)$ the set of all homomorphisms $V_I \to V_I$. One has easily [14]:

- $A \in \mathrm{Hom}(V_I)$ if and only if $A^* \in \mathrm{Hom}(V_I)$.
- The product of any number of homomorphisms is defined and is a homomorphism.
- If B_0 and B_{n+1} are arbitrary operators and if $A_1, \ldots, A_n$ are homomorphisms, then the product of $n+2$ factors $B_0 A_1, \ldots, A_n B_{n+1}$ is defined.
- If $A \in \mathrm{Hom}(V_I)$, then $f\#g$ implies $Af\#Ag$.
- $\mathrm{Hom}(V_I) \supset \mathfrak{A}(V_I)$.
- If $A \in \mathrm{Hom}(V_I)$, then A^*A is defined and $A^*A \in \mathfrak{A}(V_I)$.

Note that we can now also define an orthogonal projection in V_I as a homomorphism that satisfies the relations $P^2 = P = P^*$.

Similar definitions may be given for $\mathrm{Hom}(V_I, Y_K)$, the set of homomorphisms between two LHS V_I, Y_K. Of course, if $A \in \mathrm{Hom}(V_I, Y_K)$, then $A^* \in \mathrm{Hom}(Y_K, V_I)$.

An operator A is an *isomorphism* if $A \in \mathrm{Hom}(V_I)$ and there is a homomorphism $B \in \mathrm{Hom}(V_I)$ such that $AB = BA = \mathbb{I}$, the identity operator.

An operator U is *unitary* if U^*U and UU^* are defined and $U^*U = UU^* = \mathbb{I}$. We emphasize that a unitary operator need *not* be a homomorphism. In fact, as we will see now, *unitary isomorphisms* are the natural setting for group representations in a LHS.

5. Realization of symmetries in PIP-spaces

Assume now that the state space of a given quantum system is a (nondegenerate, positive definite) PIP-space V_I, with central Hilbert space $\mathfrak{H}_o$.

Suppose the system has a *symmetry group* G. According to Wigner and Bargmann (see section 1.1), there exists a unitary representation U_o of G in $\mathfrak{H}_o$:

$$U_o U_o^* = U_o^* U_o = \mathbb{I}_o, \qquad \text{the identity operator in } \mathfrak{H}_o.$$

Therefore, U_o must extend to a unitary representation U in V_I.

In virtue of the conservation of transition amplitudes, if $\psi_2 \# \psi_1$, one must have $U(g)\psi_2 \# U(g)\psi_1, \forall\, g \in G$, and

$$\langle U(g)\psi_2 | U(g)\psi_1\rangle = \langle \psi_2 | \psi_1\rangle.$$

Since $U(g)^* = U(g^{-1})$, this implies that $U(g)$ must be *totally regular*, $\forall\, g \in G$. Thus, one must require that U is a representation of G by *unitary, totally regular automorphisms* of V_I.

To give a simple example, let V_I be the scale built on the powers of the Hamiltonian $H = -\Delta + V(\mathbf{r})$, with $V(\mathbf{r})$ a (nice) rotation invariant potential. The system admits as symmetry group $G = \mathrm{SO}(3)$ and the representation U_o is the natural representation of SO(3) in $L^2(\mathbb{R}^3)$:

$$[U_o(R)\psi](\mathbf{x}) = \psi(R^{-1}\mathbf{x}), \qquad R \in \mathrm{SO}(3).$$

Then one has:

- U_o extends to a unitary representation U by totally regular automorphisms of V_I.
- Angular momentum decompositions extend to V_I as well.
- This is a good setting also for representations of the Lie algebra $\mathfrak{so}(3)$.

The same analysis applies to internal symmetries, if any, and discrete symmetries, such as space or time reflection.

6. Outcome

PIP-spaces yield a framework suitable for the description of quantum systems and their symmetry properties. They generalize both the traditional Hilbert space approach and the rigged Hilbert space approach, yet the mathematics involved is simpler, there is no need for sophisticated functional analysis concepts. Clearly, much work remains to be done, more realistic examples should be analyzed. Of particular interest are, of course, singular or infinite dimensional systems.

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